ALGEBRAIC TOPOLOGY I WS23/24, HOMEWORK SHEET 1

DEADLINE: FRIDAY, OCTOBER 20TH

Problem 1

The goal of the first problem is to recall the notion of a Serre fibration and its homotopical properties.

- A map of spaces $p : E \to B$ has the *homotopy lifting property with respect to a space* X if for every commutative diagram of the form
$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\widetilde{f}_0} & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{f} & B \end{array}$$
there exists a map $\widetilde{f} : X \times I \to E$ making the diagram commute ($I = [0, 1]$ being the interval).
- A map $p : E \to B$ has the *homotopy lifting property with respect to a pair of spaces* (X, A) if for every commutative square of the form
$$\begin{array}{ccc} X \cup_A (A \times I) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{f} & B \end{array}$$
there exists a map $\widetilde{f} : X \times I \to E$ making the diagram commute. (The space $X \cup_A (A \times I)$ is defined by gluing $A \times I$ to X along the natural map $A \times \{0\} \to X$.)
- A map of spaces $p : E \to B$ is said to be a *Serre fibration* if it has the homotopy lifting property with respect to all discs D^n, $n \geq 0$. It can be shown that having the homotopy lifting property with respect to all discs is equivalent to having the homotopy lifting property with respect to all CW-pairs.

Furthermore, we recall the notion of a *homotopy fibre*. For a space X and $x \in X$ we let $P_x X$ denote the space of paths in X to x, that is $P_x X := \{\gamma : [0, 1] \to X \mid \gamma(1) = x\}$, equipped with the compact-open topology. Given a map $f \colon Y \to X$, the *homotopy fibre of* $f : Y \to X$ *at* x is then defined as the space

$$\mathrm{hofib}_x(f) := P_x X \times_X Y = \{(\gamma, y) \in P_x \times Y \mid \gamma(0) = f(y)\}.$$

Now let $p : E \to B$ be a Serre fibration and $b \in B$ a basepoint. We write $F = p^{-1}(b) \subseteq E$ for the fibre, and define a map $\varphi : F \to \mathrm{hofib}_b(p)$ by the formula

$$z \mapsto (c(b), i(z)).$$

Here, $c(b)$ denotes the constant path in B at the basepoint b.
Task: Prove that φ is a weak homotopy equivalence, i.e., that it induces an isomorphism on homotopy groups for all basepoints.

(Hint: If you have trouble with the proof, first focus on showing that φ induces a bijection on path components.)

Problem 2

Let C be a chain complex, filtered by subcomplexes $C_0 \subseteq C_1 \subseteq C$. The pairs (C_1, C_0) and (C, C_1) have associated long exact sequences of homology groups

$$\cdots \to H_n(C_0) \to H_n(C_1) \to H_n(C_1/C_0) \xrightarrow{\partial} H_{n-1}(C_0) \to \cdots$$

and

$$\cdots \to H_n(C_1) \to H_n(C) \to H_n(C/C_1) \xrightarrow{\partial} H_{n-1}(C_1) \to \cdots,$$

respectively. The goal of this exercise is to compute the homology of C in terms of the homology of the complexes C_0, C_1/C_0 and C/C_1. This is the length 2 special case of the spectral sequence associated to a filtered complex which we will later discuss in the lecture.

a. Use the long exact sequences above to define maps $f : H_*(C/C_1) \to H_{*-1}(C_1/C_0)$ and $g : H_*(C_1/C_0) \to H_{*-1}(C_0)$. Show that $g \circ f = 0$. In spectral sequence terminology, the maps g and f are the only potentially nonzero d^1-differentials.

b. Next we will construct the only potentially nonzero d^2-differential. Use the long exact sequences above once more to construct another map $d : \ker(f) \to \operatorname{coker}(g)$ of degree -1 so that there are isomorphisms

- $\operatorname{coker}(d) \cong \operatorname{im}(H_*(C_0) \longrightarrow H_*(C))$
- $\ker(g)/\operatorname{im}(f) \cong \operatorname{im}(H_*(C_1) \longrightarrow H_*(C))/\operatorname{im}(H_*(C_0) \longrightarrow H_*(C))$
- $\ker(d) \cong H_*(C)/\operatorname{im}(H_*(C_1) \longrightarrow H_*(C))$,

where $\operatorname{im}(-)$ denotes the image of a map. In words, $\operatorname{coker}(d)$, $\ker(g)/\operatorname{im}(f)$ and $\ker(d)$ are isomorphic to the subquotients in the filtration on $H_*(C)$ given by the images of $H_*(C_0)$ and $H_*(C_1)$.